

Group Theoretical Structure and Inverse Scattering Method for super-KdV equation

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Abstract

Using the group-theoretical approach to the inverse scattering method the supersymmetric Korteweg-de Vries equation is obtained by application of the Drinfeld-Sokolov reduction to $osp(1|2)$ loop superalgebra. The direct and inverse scattering problems are discussed for the corresponding Lax pair.

Introduction

This article is devoted to the study of supersymmetric extension of the Korteweg-de Vries equation (super-KdV) ([4], [5]):

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 12\varepsilon\varepsilon_{xx}, \\ \varepsilon_t &= -4\varepsilon_{xxx} + 6u\varepsilon_x + 3u_x\varepsilon, \end{aligned} \tag{1}$$

where $\varepsilon(x, t)$ is a function with values in the odd part of the Grassmann algebra $\Lambda(n)_{\bar{1}}$, and $u(x, t)$ is a function with values in its even part $\Lambda(n)_{\bar{0}}$ (see Appendix). This equation is Hamiltonian one with the following Poisson brackets:

$$\begin{aligned} \{u(x), u(y)\} &= \frac{1}{2}(\delta'''(x-y) - 2u'(x)\delta(x-y) - 4u(x)\delta'(x-y)) \\ \{u(x), \varepsilon(y)\} &= -\frac{1}{2}(3\varepsilon(x)\delta'(x-y) + \varepsilon'(x)\delta(x-y)) \\ \{\varepsilon(x), \varepsilon(y)\} &= -\frac{1}{2}(\delta''(x-y) - u(x)\delta(x-y)) \end{aligned}$$

which generate the superconformal algebra. The corresponding Hamiltonian is:

$$H = \int (u^2(x) - 4\varepsilon(x)\varepsilon'(x))dx.$$

This equation is integrable and allows the Lax representation: $\dot{\mathcal{L}} = [\mathcal{M}, \mathcal{L}]$.

The first part of this article is devoted to the mentioned above, i.e. obtaining the Poisson structure and the Lax representation for the super-KdV equation.

In the second part the application of the inverse scattering method (ISM) [6]-[8] to this system is considered. The attention is devoted mostly to the case of one-dimensional Grassmann algebra, when the super-KdV equation is reduced to the system of equations:

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x \\ e_t &= -4e_{xxx} + 6ue_x + 3u_xe \end{aligned} \tag{2}$$

where $u(x, t)$, $e(x, t)$ are some functions with values in \mathbf{R} .

This system naturally appears in the case of Grassmann algebra of any dimension, if we write down the super-KdV equation using the basis of monomials: $\Theta^{i_1}\Theta^{i_2}\dots\Theta^{i_n}$ (see Appendix). Really, restricting to the terms, linear in Θ^i , we obtain the system (2).

Moreover, it should be noted, that the only nonlinear equation (in the basis decomposition of the super-KdV equation) is the usual KdV equation, all others are linear and the whole system can be written in the “triangular form”.

One-dimensional case is interesting, because the pair of equations (2) can be expressed in such a way:

$$\begin{cases} \dot{L} = [L, A] \\ e_t = -Ae \end{cases} \tag{3}$$

where $L = -\partial_x^2 + u(x)$, $A = 4\partial_x^3 - 3(u\partial_x + \partial_x u)$ is a scalar L-A pair for usual KdV. One can easily derive that if $e(x, t)$ is a solution for given $u(x, t)$, then $L^n e, n = 1, 2, \dots$ is also a solution for this u . Thus we obtain an infinite chain of solutions, except the case when $e(x, t)$ is an eigenfunction of L . For example, if one takes the one-soliton solution of KdV:

$$u(x, t) = \frac{c}{2\text{ch}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right)} \tag{4}$$

and then supposing, that $e(x, t)$ is an eigenfunction of L , one can obtain a solution:

$$e(x, t) = \frac{a}{2\text{ch}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right)} \quad (a \in \mathbf{R}). \quad (5)$$

These solutions can be obtained also by the ISM (see part 2). Nowadays many papers are devoted to the superequations due to the progress in the superstring theory. It is reasonable to mention the papers [9]-[11]. Moreover, if we want to quantize such systems, it would be useful to study their behavior on the classical level.

1 Group-theoretical structure of the super-KdV equation

The usual KdV equation is related with the scalar Lax operator

$$L = -\partial_x^2 + \lambda + u$$

or with the equivalent matrix Lax operator

$$\partial_x + \begin{pmatrix} 0 & 1 \\ u + \lambda & 0 \end{pmatrix},$$

and the Lie algebra $sl(2)$. The Hamiltonian structure for the KdV equation is given by the Virasoro algebra:

$$\{u(x), u(y)\} = \frac{1}{2}(\delta'''(x - y) - (4u(x)\partial_x + 2u_x(x))\delta(x - y)) \quad (6)$$

The relation of these two objects with the KdV equation is given by the group-theoretical scheme and the Drinfeld-Sokolov reduction.

In this part of the article, this formalism is applied to the superalgebra $\text{osp}(1|2)$, and following this approach the super-KdV equation is obtained.

1.1 Group-theoretical structure and the Drinfeld-Sokolov reduction for $\text{osp}(1|2)$

The following constructions were applied to the Lie algebras (see, for example [1],[2],[12]). We extend them to the case of superalgebras, in particular to the $\text{osp}(1|2)$ superalgebra (see Appendix).

Let \mathcal{G} be a Lie superalgebra and let

$$A = \ell(\mathcal{G}) = \mathcal{G} \otimes C[\lambda, \lambda^{-1}] = \left\{ \sum_{i=r}^s X_i \lambda^i \mid X_i \in \mathcal{G} \right\}$$

be its affinization by the spectral parameter λ . $A = A_+ + A_-$, where $A_+ = \mathcal{G} \otimes C[\lambda] = \{\sum_{i=0}^s X_i \lambda^i \mid X_i \in \mathcal{G}\}$, $A_- = \mathcal{G} \otimes \lambda^{-1} C[\lambda^{-1}] = \{\sum_{i=r}^{-1} X_i \lambda^i \mid X_i \in \mathcal{G}\}$. Let $R = P_+ - P_-$, where P_+ and P_- are projectors on A_+ A_- correspondingly. Moreover, let us introduce multiplication operator $\hat{\eta} : A \rightarrow A$ for function $\eta(\lambda) \in C[\lambda, \lambda^{-1}]$ in such a way: $(\hat{\eta}X)(\lambda) = \eta(\lambda)X(\lambda)$. Then we define the operator $R_\eta = R \circ \hat{\eta}$. It appears that R_η defines classical R -matrix for all η , i.e.

$$[X, Y]_{R_\eta} = \frac{1}{2}[R_\eta X, Y] + \frac{1}{2}[X, R_\eta Y]$$

is also a Lie superbracket.

We can identify the dual space $A^* = \ell(\mathcal{G})^*$ with $\ell(\mathcal{G}^*) = \mathcal{G}^* \otimes C[\lambda, \lambda^{-1}]$ by means of pairing:

$$\begin{aligned} \langle \alpha, x \rangle &= \langle \sum_i \alpha_i \lambda^i, \sum_j x^j \lambda^j \rangle = \sum_{i+j=-1} \alpha_i(x_j), \\ \alpha_i &\in \mathcal{G}^*, \quad x_i \in \mathcal{G} \end{aligned} \quad (7)$$

Now we are ready to introduce the Poisson brackets on A^* (we mean structures of the Lie Poisson type). We are interested in the following Poisson brackets:

$$\{\phi, \psi\}_{R_\eta}(\alpha) = \langle \alpha, [\nabla_\alpha \phi, \nabla_\alpha \psi]_{R_\eta} \rangle \quad (8)$$

where $\phi, \psi \in C^\infty(A^*)$, $\alpha \in A^*$. If we consider the space of functions, invariant under the coadjoint action:

$$I(A^*) = \{\phi \in C^\infty(A^*) \mid \phi(Ad_g^* \alpha) = \phi(\alpha), \quad \forall g \in \ell(G)\},$$

where $\ell(G)$ is a supergroup, constructed from the Lie superalgebra $\ell(\mathcal{G})$, then the same proposition as for usual Lie algebras takes place, i.e.

$$\{\phi, \psi\}_R(\alpha) = 0, \quad \alpha \in A^*, \phi, \psi \in I(A^*)$$

Now let's consider the central extension of our algebra. We will call \mathcal{G}_c the linear space of pairs $\mathcal{G}_c = \{(X, a) \mid X : S^1 \rightarrow \mathcal{G}; a \in \mathbf{C}\}$, with the following bracket:

$$[(X, a), (Y, b)] = (XY - YX, \text{str} \int_{S^1} dx X'(x) Y(x)).$$

One can obtain the isomorphism between \mathcal{G}_c^* and the space of operators $\mathcal{L} = e\partial_x + \mu$, where $(\mu, e) \in \mathcal{G}_c^*$. The pairing here is defined as follows:

$\langle (\mu, e), (X, a) \rangle = ea + str \int \mu X$. Using the previous discussion, we can introduce the affinization of \mathcal{G}_c by means of spectral parameter λ and obtain in such a way algebra $A^* = \ell(\mathcal{G}_c^*)$. From the identification of $\ell(\mathcal{G}_c^*)$ with the differential operators one can easily see that the coadjoint representation is given by the “nonabelian gauge transformation” [1]:

$$\mathcal{L} = e\partial_x + \mu \rightarrow g\mathcal{L}g^{-1} = e\partial_x + g\mu g^{-1} - eg'g^{-1},$$

where $g \in \ell(G)$. Consider the space $H_C(\mathcal{G})$, consisting of the elements of the following type: $(\mu, e)(\lambda) = (J(x) + C\lambda, e_0 + e_1\lambda)$. Let $\eta(\lambda) = \eta_0 + \eta_1\lambda$. Then R_η Poisson bracket can be expressed in such a way:

$$\begin{aligned} \{\phi, \psi\}_{R_\eta}(J) = & -\eta_0 \left(str \int C \left[\frac{\delta\phi}{\delta J(z)}, \frac{\delta\psi}{\delta J(z)} \right] \right. \\ & \left. + e_1 str \int \partial_z \left(\frac{\delta\phi}{\delta J(z)} \right) \frac{\delta\psi}{\delta J(z)} \right) + \\ & + \eta_1 \left(str \int J(z) \left[\frac{\delta\phi}{\delta J(z)}, \frac{\delta\psi}{\delta J(z)} \right] + e_0 str \int \partial_z \left(\frac{\delta\phi}{\delta J(z)} \right) \frac{\delta\psi}{\delta J(z)} \right) \end{aligned}$$

Let now $e_0 = 1, e_1 = 0$. Considering two cases: $\eta_1 = 0$ or $\eta_0 = 0$ we obtain two types of Poisson brackets:

$$\begin{aligned} \{\phi, \psi\}_1(J) &= -str \int C \left[\frac{\delta\phi}{\delta J(z)}, \frac{\delta\psi}{\delta J(z)} \right] \\ \{\phi, \psi\}_2(J) &= str \int J(z) \left[\frac{\delta\phi}{\delta J(z)}, \frac{\delta\psi}{\delta J(z)} \right] + str \int \partial_z \left(\frac{\delta\phi}{\delta J(z)} \right) \frac{\delta\psi}{\delta J(z)} \end{aligned} \tag{9}$$

So, let us summarize, what was obtained. Two Poisson brackets (9) are defined on the space $H_C(\mathcal{G})$ of the following operators:

$$\partial_x + J(x) + \lambda C. \tag{10}$$

These Hamiltonian structures are invariant under the mentioned above non-abelian gauge transformations of the operators (10), for which the following condition is satisfied: $gCg^{-1} = C, g \in G$.

Now let us move to the $\text{osp}(1|2)$ case. One can reduce $H_C(\text{osp}(1|2))$ to the subspace $H_C^{\text{constr}}(\text{osp}(1|2)) = \{\mathcal{L} = \partial_x + \alpha(x)v_- + w(x)X_- + q(x)h + \beta(x)v_+ + \lambda X_- + X_+, \text{ where } \alpha(x) \text{ is an odd element of Grassmann algebra and } q(x), w(x) \text{ are even ones}\}$. Note, that we have put $C = X_-$. The matrix form \mathcal{L} is:

$$\mathcal{L} = \partial_x + \begin{pmatrix} q(x) & \beta(x) & 1 \\ -\alpha(x) & 0 & \beta(x) \\ \lambda + w(x) & \alpha(x) & -q(x) \end{pmatrix} \tag{11}$$

Maximal group of gauge transformations, preserving $X_- = C$, and therefore the form of operator \mathcal{L} and the brackets (9) is $G_- = \{\exp(\beta(x)v_- + p(x)X_-)\}$ where $\beta(x)$ - odd and $p(x)$ - even. One can consider the factorspace $H_{X_-}^{constr}(\mathfrak{osp}(1|2))/G_- = H_{X_-}^{red}(\mathfrak{osp}(1|2))$. We obtain the Hamiltonian projection $\pi : H_{X_-}^{constr} \rightarrow H_{X_-}^{red}$. It is Hamiltonian in the sense, that π preserves brackets (9).

Hamiltonian reduction of such a type is called the Drinfeld-Sokolov reduction for the superalgebra $\mathfrak{osp}(1|2)$ [12]. Every point of the manifold $H_{X_-}^{red}(\mathfrak{osp}(1|2))$ corresponds to the differential operator

$$\mathcal{L} = \partial_x + \begin{pmatrix} 0 & 0 & 1 \\ -\varepsilon(x) & 0 & 0 \\ u(x) + \lambda & \varepsilon(x) & 0 \end{pmatrix} \quad (12)$$

where $u(x), \varepsilon(x)$ are even and odd elements of Grassmann algebra correspondingly. Really, we have taken one element from each orbit of the group G_- , acting on $H_{X_-}^{constr}(\mathfrak{osp}(1|2))$. Now, let's consider the linear problem $\mathcal{L}\Psi = 0$, where

$$\Psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix}$$

with $\psi_1(x), \psi_3(x)$ lying in the even part of Grassmann algebra and $\psi_2(x)$ lying in the odd part. This problem reduces to the scalar one:

$$L\psi_1(x) := (-\partial_x^2 + \lambda + u(x) - \varepsilon(x)\partial^{-1}\varepsilon(x))\psi_1(x, \lambda) = 0 \quad (13)$$

If we consider the problem $\mathcal{L}'\Psi = 0$, $g\mathcal{L}g^{-1} = \mathcal{L}'$, we obtain (13) again, because ψ_1 is invariant under the transformations of G_- element. Really,

$$\begin{aligned} g^{-1}\Psi &= \begin{pmatrix} 1 & & \\ -\beta(x) & 1 & \\ p(x) & \beta(x) & 1 \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_1(x) \\ \tilde{\psi}_2(x) \\ \tilde{\psi}_3(x) \end{pmatrix} = \tilde{\Psi} \quad (14) \\ \Rightarrow g\mathcal{L}g^{-1}\Psi &= 0 \Leftrightarrow \mathcal{L}\tilde{\Psi} = 0 \Leftrightarrow (13) \end{aligned}$$

It seems that $H_{X_-}^{red}(\mathfrak{osp}(1|2))$ can be identified with operators $L = -\partial_x^2 + \lambda + u(x) - \varepsilon(x)\partial^{-1}\varepsilon(x)$. But we can't do this because we can lose the part of the solutions of the matrix linear problem $\mathcal{L}\Psi = 0$. For example, in the case of 1-dimensional Grassmann algebra operator L coincides with the Sturm-Liouville operator and does not take into account the solution of the following form:

$$\Psi = \begin{pmatrix} 0 \\ \theta c \\ 0 \end{pmatrix}, \quad (15)$$

where c is a constant and θ is an odd basis element of Grassmann algebra. Let's now rewrite the Poisson brackets (9) on the manifold $H_{X_-}^{red}(\text{osp}(1/2))$ in terms of its parameters $u(x), \varepsilon(x)$:

$$\begin{aligned}\{u(x), u(y)\}_2 &= \frac{1}{2}(\delta'''(x-y) - 2u'(x)\delta(x-y) - 4u(x)\delta'(x-y)) \\ \{u(x), \varepsilon(y)\}_2 &= -\frac{1}{2}(3\varepsilon(x)\delta'(x-y) + \varepsilon'(x)\delta(x-y)) \\ \{\varepsilon(x), \varepsilon(y)\}_2 &= -\frac{1}{2}(\delta''(x-y) - u(x)\delta(x-y))\end{aligned}\tag{16}$$

$$\begin{aligned}\{u(x), u(y)\}_1 &= -2\delta'(x-y) \\ \{u(x), \varepsilon(y)\}_1 &= 0 \\ \{\varepsilon(x), \varepsilon(y)\}_1 &= \frac{1}{2}\delta(x-y)\end{aligned}\tag{17}$$

Thus, u, ε generate the superconformal algebra with the brackets (16). Poisson brackets (17) are the linearization of (16) .

1.2 Infinite family of Hamiltonians. The super-KdV equation.

We can rewrite (13) in such a way:

$$(-\partial_x^2 + u(x) - \varepsilon(x)\partial^{-1}\varepsilon(x))\psi(x, k) = k^2\psi(x, k) \quad (k^2 = -\lambda).\tag{18}$$

Using the following ansatz:

$$\psi(x, k) = \exp(-ikx + \sum_{n=0}^{\infty} f_n(x)(2ik)^{-n})$$

substituting it in (18), we obtain the set of equations on $f_n(x)$. Changing ∂^{-1} by \int_0^x and $f_n(x)$ by $\int_0^x \sigma_n(y)dy$, we obtain that $\int_0^{2\pi} \sigma_n(y)dy = H_n(\mathcal{L})$ - form an involutive family of gauge invariant functionals of \mathcal{L} . Here are the first three of them:

$$\begin{aligned}H_1 &= \int u(x)dx \\ H_2 &= \int (u^2(x) - 4\varepsilon(x)\varepsilon'(x))dx \\ H_3 &= \int ((u')^2 + 2u^3 - 16\varepsilon'(x)\varepsilon''(x) - 24\varepsilon(x)\varepsilon'(x)u(x))dx\end{aligned}$$

From a Drinfeld-Sokolov theory we obtain the following property:

$$\{H_{i+1}, \phi\}_1 = \{H_i, \phi\}_2.$$

Now, let's write down equations of motion for $u(x, t)$, $\varepsilon(x, t)$ with the Hamiltonian H_2 and the bracket $\{, \}_2$:

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 12\varepsilon\varepsilon_{xx}, \\ \varepsilon_t &= -4\varepsilon_{xxx} + 6u\varepsilon_x + 3u_x\varepsilon. \end{aligned} \quad (19)$$

System of equations (19) is called a super-KdV equation. At first it was obtained in [4], [5]. Moreover, super-KdV system can be obtained from the \mathcal{L} , \mathcal{M} pair, that is, as a compatibility conditions for the two equations:

$$\begin{aligned} \mathcal{L}\psi &= 0 \\ \partial_t\psi &= \mathcal{M}\psi \end{aligned}$$

where

$$\mathcal{M} = \begin{pmatrix} -u_x & -4\varepsilon_x & 4\lambda - 2u \\ 2\varepsilon u - 4\varepsilon\lambda - 4\varepsilon_{xx} & 0 & -4\varepsilon_x \\ 4\varepsilon\varepsilon_x + u_{xx} - 2u^2 + 2u\lambda + 4\lambda^2 & -2\varepsilon u + 4\varepsilon\lambda + 4\varepsilon_{xx} & u_x \end{pmatrix}. \quad (20)$$

2 Inverse Scattering Method for the super-KdV equation

In this part of the article the super-KdV equation is considered from a point of view of the inverse scattering method. At first the direct scattering problem is analyzed and the relations between the elements of transfer matrix $T(k)$ are found. Then, in the case of 1-dimensional Grassmann algebra we consider the inverse problem. At the end the explicit solitonic solutions are introduced.

2.1 Direct Problem

The super-KdV equation can be obtained as a compatibility condition for two equations:

$$\begin{aligned} \mathcal{L}\psi &= 0 \\ \partial_t\psi &= \mathcal{M}\psi \end{aligned}$$

In this subsection we analyze the linear problem $\mathcal{L}\psi = 0$. Let's transform it. First of all we consider the group element $U = \exp(ikX_-)$ and using it we make a similarity transformation $U\mathcal{L}U^{-1}$, thus obtaining the operator:

$$\mathcal{L}' = \partial_x + \begin{pmatrix} -ik & 0 & 1 \\ -\varepsilon & 0 & 0 \\ u & \varepsilon & ik \end{pmatrix} \quad (21)$$

Then, using the matrix N :

$$N = \begin{pmatrix} 1 & 0 & -(2ik)^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & (2ik)^{-1} \end{pmatrix} \quad (22)$$

one can bring the constant part of \mathcal{L}' to the diagonal form:

$$\tilde{\mathcal{L}}(x) = \begin{pmatrix} -u(2ik)^{-1} & -\varepsilon(2ik)^{-1} & -u(2ik)^{-1} \\ -\varepsilon & 0 & -\varepsilon \\ u(2ik)^{-1} & \varepsilon(2ik)^{-1} & u(2ik)^{-1} \end{pmatrix} + \begin{pmatrix} -ik & & \\ & 0 & \\ & & ik \end{pmatrix} + \partial_x$$

We can rewrite the corresponding linear problem as follows:

$$\partial_x \Psi = ikh\Psi + Q(x, k)\Psi,$$

where

$$Q(x, k) = \begin{pmatrix} u(2ik)^{-1} & \varepsilon(2ik)^{-1} & u(2ik)^{-1} \\ \varepsilon & 0 & \varepsilon \\ -u(2ik)^{-1} & -\varepsilon(2ik)^{-1} & -u(2ik)^{-1} \end{pmatrix} \quad (23)$$

Let's consider the following matrix solutions (Jost solutions):

$$\begin{aligned} \Phi^+(x, k) &= \begin{pmatrix} e^{ikx} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-ikx} \end{pmatrix} & x \rightarrow \infty, \\ \Phi^-(x, k) &= \begin{pmatrix} e^{ikx} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-ikx} \end{pmatrix} & x \rightarrow -\infty \end{aligned} \quad (24)$$

This matrix valued functions are constructed in such a way that $N^{-1}\Phi^\pm N$ are the elements of the group $\text{Osp}(1|2)$. Let $\Phi^+(x, k)T(k) = \Phi^-(x, k)$, where

$$T(k) = \begin{pmatrix} a(k) & \gamma(k) & b(k) \\ \xi(k) & f(k) & \delta(k) \\ c(k) & \eta(k) & d(k) \end{pmatrix} \quad (25)$$

is a transfer matrix. One can find relations between elements of $T(k)$. Really, defining the matrix P :

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (26)$$

it is easy to see, that if $\Psi(x, k)$ is a solution of (23), then $P\Psi(x, k)P = \tilde{\Psi}(x, k)$ satisfies the following equation:

$$\partial_x P\Psi P = -ikhP\Psi P + Q^*P\Psi P$$

where $*$ means the reversing of the sign of k (if $k \in \mathbf{R}$, then it is the complex conjugation):

$$\tilde{\Psi}(x, k) = \Psi(x, -k) = \Psi^*(x, k).$$

Thus $T^* = PTP$ and we can reduce the transfer matrix [7],[8]:

$$T(k) = \begin{pmatrix} a & \gamma & b \\ \xi & f & \delta \\ c & \eta & d \end{pmatrix} = \begin{pmatrix} a & \gamma & b \\ \bar{\delta} & f & \delta \\ \bar{b} & \bar{\gamma} & \bar{a} \end{pmatrix} \quad (27)$$

that is $\xi = \bar{\delta}$, $d = \bar{a}$, $\eta = \bar{\gamma}$, $f = \bar{f}$ (bar means the same as $*$). Moreover, we know that $N^{-1}TN \in \text{Osp}(1|2)$. In such a way we have found the constraints:

$$\begin{aligned} f &= 1 + 2ik\bar{\gamma}\gamma \\ f(a\bar{a} - b\bar{b}) &= 1 \\ \delta &= 2ik(\bar{\gamma}b - \gamma\bar{a}) \end{aligned}$$

In the case of 1-dimensional Grassmann algebra they have the following form:

$$f = 1, \quad a\bar{a} - b\bar{b} = 1, \quad \delta = 2ik(\bar{\gamma}b - \gamma\bar{a})$$

After this we consider the factorization of the matrix $T(k)$: $T^+ = TT^-$, where

$$T^+(k) = \begin{pmatrix} 1 & -\delta(2ik)^{-1} & b \\ 0 & \bar{a} & \delta \\ 0 & 0 & \bar{a} \end{pmatrix}, \quad T^-(k) = \begin{pmatrix} \bar{a} & 0 & 0 \\ -(2ik)\bar{\gamma} & \bar{a} & 0 \\ -\bar{b} & -\bar{\gamma} & 1 \end{pmatrix} \quad (28)$$

One can construct the matrix-valued function:

$$\Psi^+(x, k) = \Phi^+(x, k)T^+(k)e^{-ikxh} = \Phi^-(x, k)T^-(k)e^{-ikxh} \quad (29)$$

Asymptotics of it's diagonal components have the folowing expression:

$$\pi_0 T^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{a} & 0 \\ 0 & 0 & \bar{a} \end{pmatrix} \quad (30)$$

$$x \rightarrow \infty$$

$$\pi_0 T^- = \begin{pmatrix} \bar{a} & 0 & 0 \\ 0 & \bar{a} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (31)$$

$x \rightarrow -\infty$

$\Psi^+(x, k)$ satisfies the integral equation:

$$\begin{aligned} \Psi^+(x, k) = \pi_0 T^+ &+ \int_{-\infty}^x e^{ikad_h(x-y)} \pi_+(Q(y, k) \Psi^+(y, k)) dy \\ &- \int_x^\infty e^{ikad_h(x-y)} (\pi_0 + \pi_-)(Q(y, k) \Psi^+(y, k)) dy \end{aligned} \quad (32)$$

where π_0 is the projection on the diagonal part of the corresponding matrix, and π_\pm are the projections on the strict upper triangular and strict lower triangular parts correspondingly. Let's write the integral equation for the first column of $\Psi^+(x, k)$:

$$\Psi^+(x, k) = \begin{pmatrix} \Psi_{11} \\ \Psi_{21} \\ \Psi_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -(2ik)^{-1} \int_x^\infty ((\Psi_{11}(y, k) + \Psi_{31}(y, k))u(y) - \varepsilon(y)\Psi_{21}(y, k)) dy \\ - \int_x^\infty e^{-ik(x-y)} (\varepsilon(y)(\Psi_{11}(y, k) + \Psi_{31}(y, k))) dy \\ (2ik)^{-1} \int_x^\infty e^{-2ik(x-y)} ((\Psi_{11}(y, k) + \Psi_{31}(y, k))u(y) - \varepsilon(y)\Psi_{21}(y, k)) dy \end{pmatrix}.$$

We derive from this the relation for the first column of the matrix Jost solution $\Phi^{+(1)}(x, k) = \Psi^{+(1)}(x, k)e^{ikx}$ and, correspondingly,

$$N^{-1}\Psi^{+(1)}(x, k) = \tilde{m}_+(x, k),$$

where

$$\tilde{m}_+(x, k) = \begin{pmatrix} 1 + \int_x^\infty dy \frac{e^{-2ik(x-y)} - 1}{2ik} (\tilde{m}_+^1(y, k)u(y) - \epsilon(y)\tilde{m}_+^2(y, k)) \\ - \int_x^\infty dy e^{-ik(x-y)} \epsilon(y)\tilde{m}_+^1(y, k) \\ \int_x^\infty dy e^{-2ik(x-y)} (u(y)\tilde{m}_+^1(y, k) - \epsilon(y)\tilde{m}_+^2(y, k)) \end{pmatrix}. \quad (33)$$

Now we can see, that $\tilde{m}_+(x, k)$ allows analytical continuation in the upper half-plane of k . Analogously, considering factorization $R^- = TR^+$ and corresponding integral equations, one can prove that $\Phi^{-(1)}(x, k)e^{-ikx}$ is analytic in the lower half-plane. From T^+, T^- factorization we obtain that

$$\begin{aligned} \Phi^{+(1)}(x, k)e^{-ikx} &= \bar{a}(k)\Phi^{-(1)}(x, k)e^{-ikx} - 2ik\bar{\gamma}(k)\Phi^{-(2)}(x, k)e^{-ikx} \\ &- \bar{b}\Phi^{-(3)}(x, k)e^{-ikx} \end{aligned} \quad (34)$$

Using the above reasoning, one can rewrite it in the following way:

$$\begin{aligned} m_+(x, k) - m_-(x, k) &= -2ik\rho(k)N^{-1}(k)\Phi^{-(2)}(x, k)e^{-ikx} \\ &- r(k)e^{-2ikx}P(k)m_-(x, -k) \end{aligned} \quad (35)$$

where

$$\begin{aligned} m_+(x, k) &= \frac{\tilde{m}_+(x, k)}{\bar{a}(k)}, \quad m_-(x, k) = \tilde{m}_-(x, k), \quad \rho(k) = \frac{\bar{\gamma}(k)}{\bar{a}(k)}, \\ r(k) &= \frac{\bar{b}(k)}{\bar{a}(k)}, \quad P(k) = N^{-1}(k)PN(-k) \end{aligned}$$

or :

$$m_+(x, k) - m_-(x, k) = V(k, x)m_-(x, -k) + f(x, k) \quad (36)$$

where

$$V(k, x) = -r(k)e^{-2ikx}P(k), \quad f(x, k) = -2ik\rho(k)N^{-1}(k)\Phi^{-2}(x, k)e^{-ikx}$$

Moreover, the equation (33) gives:

$$u(x) = 2ik\partial_x\tilde{m}_+^1(x, k), \quad \varepsilon(x) = ik\tilde{m}_+^2(x, k), \quad |k| \rightarrow \infty \quad (37)$$

Thus, we can restore $u(x)$, $\varepsilon(x)$ by means of $r(k)$, $\rho(k)$, solving (36). However, in the general case we can say nothing about the behaviour of $\Phi^{-(2)}(x, k)$, so, from now on, we study the case of 1-dimensional Grassmann algebra, for which

$$\rho(k)\Phi^{-(2)}(x, k) = \begin{pmatrix} 0 \\ \rho(k) \\ 0 \end{pmatrix}$$

Also, in 1-dimensional case all even elements satisfy the usual KdV properties. Variable k may be situated whether on the real line or in the discrete set of points on the imaginary axis. This happens because of the fact, that in 1-dimensional case our \mathcal{L} problem reduces to the Sturm-Liouville problem, for which, in the case of fast decreasing potential we have the continuous spectrum: positive real axis and the discrete spectrum: the set of points on the negative real axis (see [7],[8]). It is known that $a(k)$ has the following asymptotics:

$$a(k) = 1 + O\left(\frac{1}{k}\right) \quad |k| \rightarrow \infty,$$

$a(k)$ is analytic in the lower half-plane and $\bar{a}(k) = a(-k)$ in the upper one. Points of the discrete spectrum corresponds to zeros of $a(k)$ (correspondingly $\bar{a}(k)$) so, for any point of discrete spectrum $i\kappa_j$ there exist simple zero at this

point of $a(k)$ ($\bar{a}(k)$) and if $i\kappa_j$ is situated in the upper half-plane, this point corresponds to the simple pole of $m_+(x, k)$.

So, if we know the so-called “scattering data”:

$$\{i\kappa_\ell\}, \{b_\ell\}, \{\gamma_\ell\}, \rho(k), r(k)$$

we can restore $u(x), \varepsilon(x)$, solving the Riemann problem (36).

2.2 Inverse problem

Let's summarise the results obtained from study of the direct problem:

1. The scattering data:

$$\{i\kappa_\ell\}, \{b_\ell\}, \{\gamma_\ell\}, \rho(k), r(k)$$

2. Riemann problem:

$$m_+(x, k) - m_-(x, k) = V(x, k)m_-(x, k) + f(x, k), \quad (38)$$

where $m_+(x, k)$ is meromorphic in the upper half-plane with simple poles at the points $\{i\kappa_\ell\}$, $m_-(x, k)$ is analytic in the lower half-plane with the symptotics: $m_\pm(x, k) \rightarrow (1, 0, 0)^t$ when $|k| \rightarrow \infty$

$$\begin{aligned} V(x, k) &= -r(k)e^{-2ikx}N^{-1}(k)PN(-k), \\ f(x, k) &= \begin{pmatrix} 0 \\ -2ik\rho(k)e^{-2ikx} \\ 0 \end{pmatrix} \end{aligned} \quad (39)$$

The relation (38) is written for $k \in \mathbf{R}$.

We want to find $m_+(x, k)$, and then, using (37), we obtain $u(x), \varepsilon(x)$ by the formulas:

$$u(x) = 2ik\partial_x m_+^1(x, k), \quad \varepsilon(x) = ikm_+^2(x, k), \quad k \rightarrow \infty$$

The solution of (38) has the form [9]:

$$\begin{aligned} m(x, k) &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sum_{j=1}^N \frac{m_j(x)}{k - i\kappa_j} \\ &+ \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dz \frac{V(x, z)m_-(-z, x) + f(x, z)}{z - k} \end{aligned} \quad (40)$$

$m(x, k)$ is a function, which coincides with $m_-(x, k)$ in the lower half-plane; in the upper half-plane it is equal to $m_+(x, k)$ and on the real axis it has a jump (38). Moreover,

$$m_\ell(x) = \frac{\Phi^{+1}(x, i\kappa_\ell)}{\bar{a}'(i\kappa_\ell)} e^{\kappa_\ell x}$$

(see subsection 1). Let's consider the relation (34) at the point $i\kappa_n$:

$$\Phi^{+(1)}(x, i\kappa_n) e^{\kappa_n x} = 2\kappa_n \bar{\gamma}_n \Phi^{-(2)}(x, i\kappa_n) e^{\kappa_n x} - \bar{b}_n P(i\kappa_n) \Phi^{-(1)}(x, -i\kappa_n) e^{2\kappa_n x} \quad (41)$$

We deduce:

$$m_n(x) = \frac{2\kappa_n \bar{\gamma}_n e^{\kappa_n x}}{\bar{a}'(i\kappa_n)} \Phi^{-2}(x, i\kappa_n) - \frac{\bar{b}_n e^{2\kappa_n x}}{\bar{a}'(i\kappa_n)} P(i\kappa_n) \cdot \quad (42)$$

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i \sum_{j=1}^N \frac{m_j(x)}{\kappa_n + \kappa_j} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dz \frac{V(x, z) m_-(x, -z) + f(x, z)}{z - i\kappa_n} \right)$$

In such a way we obtain the following equations for the first two components of $m(x, k)$ (we denote them $R(x, k)$ and $\Theta(x, k)$):

$$\begin{aligned} R_n(x) &= -\frac{\bar{b}_n e^{2\kappa_n x}}{\bar{a}'(i\kappa_n)} \left(1 + i \sum_{j=1}^N \frac{R_j(x)}{\kappa_n + \kappa_j} - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dz \frac{r(z) R(x, -z) e^{-2izx}}{z - i\kappa_n} \right) \\ R(x) &= 1 + \sum_{j=1}^N \frac{R_j(x)}{k - i\kappa_j} - \int_{-\infty}^{+\infty} dz \frac{r(z) R(x, -z) e^{-2izx}}{z - (k + i0)} \\ \Theta_n(x) &= \frac{2\kappa_n \bar{\gamma}_n e^{\kappa_n x}}{\bar{a}'(i\kappa_n)} - \frac{\bar{b}_n e^{2\kappa_n x}}{\bar{a}'(i\kappa_n)} \left(i \sum_{j=1}^N \frac{\Theta_j(x)}{\kappa_n + \kappa_j} \right. \\ &\quad \left. - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dz \frac{r(z) \Theta(x, -z) e^{-2izx} + 2iz\rho(z) e^{-izx}}{z - i\kappa_n} \right) \\ \Theta(x) &= \sum_{j=1}^N \frac{\Theta_j(x)}{k - i\kappa_j} - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dz \frac{r(z) \Theta(x, -z) e^{-2izx} + 2iz\rho(z) e^{-izx}}{z - (k + i0)} \end{aligned} \quad (43)$$

Using these formulas we obtain the expressions for the potentials:

$$\begin{aligned} u(x) &= \partial_x \left(2i \sum_{j=1}^N R_j(x) + \frac{1}{\pi} \int_{-\infty}^{+\infty} dz (r(z) R(x, -z) e^{-2izx}) \right) \\ \varepsilon(x) &= i \sum_{j=1}^N \Theta_j(x) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz (r(z) \Theta(x, -z) e^{-2izx} + 2iz\rho(z) e^{-izx}) \end{aligned} \quad (44)$$

Now, let's recall the equation $\partial_t \Psi = \mathcal{M} \Psi$ from the beginning of subsection 1. One can include the dependence on t in T , using the relation $\Phi^- = \Phi^+ T$ with the x -asymptotics of Φ^- , Φ^+ unchanged, that is:

$$T(k, t) = e^{4ik^3 t h} T(k) e^{-4ik^3 t h}.$$

Then, the equations describing the dynamics of coefficients of transfer matrix can be written as follows:

$$\begin{aligned} \dot{\bar{a}}(k, t) &= 0 \\ \dot{\bar{\gamma}}(k, t) &= -4ik^3 \bar{\gamma}(k, t) \\ \dot{\bar{b}}(k, t) &= -8ik^3 \bar{b}(k, t) \end{aligned}$$

The expressions giving the potentials $u(x, t)$, $\varepsilon(x, t)$ coincide with (44), with the included dynamics of transfer matrix coefficients. The equations (43) are explicitly solvable in the case of reflectionless potentials, when $b(k) = 0$ and $\rho(k) = 0$. In this case the integral equations reduce to the algebraic ones:

$$\begin{aligned} R_n(x) &= -\frac{\bar{b}_n e^{2\kappa_n x}}{\bar{a}'(i\kappa_n)} \left(1 + i \sum_{j=1}^N \frac{R_j(x)}{\kappa_n + \kappa_j} \right) \\ \Theta_n(x) &= \frac{2\kappa_n \bar{\gamma}_n e^{\kappa_n x}}{\bar{a}'(i\kappa_n)} - i \frac{\bar{b}_n e^{2\kappa_n x}}{\bar{a}'(i\kappa_n)} \sum_{j=1}^N \frac{\Theta_j(x)}{\kappa_n + \kappa_j} \\ u(x) &= 2i \partial_x \sum_{j=1}^N R_j(x) \\ \varepsilon(x) &= i \sum_{j=1}^N \Theta_j(x) \end{aligned}$$

As an example one can calculate 1-soliton solution (only one point of the discrete spectrum $\lambda_\kappa = \kappa^2$). We obtain:

$$u(x, t) = -\frac{c}{2ch^2 \left(\frac{\sqrt{c}}{2}(x - ct) \right)}, \quad \varepsilon(x, t) = -\frac{\alpha}{ch \left(\frac{\sqrt{c}}{2}(x - ct) \right)} \quad (45)$$

where $c = 4\kappa^2$, $\bar{b}_1(0) = -1$ and α is any odd constant element of the Grassmann algebra.

Actually, using ISM, we can solve KdV by means of implicit changing of variables $u(x)$, $\varepsilon(x) \rightarrow s(0)$ (the scattering data). In terms of new variables the equations of motion $\dot{s} = F(s)$ are trivially solvable differential equations. Inverse transformation

$$s(t) \rightarrow \varepsilon(x, t), u(x, t)$$

with the use of the integral equations (43),(44) and the evolution of the scattering data gives the solution for super-KdV system. Thus, here is the scheme for solving the Cauchy problem [6]-[8]:

$$\begin{array}{ccccccc} & & \text{I} & & \text{II} & & \text{III} \\ \varepsilon(x, 0), u(x, 0) & \rightarrow & s(0) & \rightarrow & s(t) & \rightarrow & u(x, t), \varepsilon(x, t) \end{array}$$

At first we obtain from $u(x, 0), \varepsilon(x, 0)$ two functions $\rho(k), r(k)$ on the half-line (by virtue of relations $\bar{r}(k) = r(-k), \bar{\rho}(k) = \rho(-k)$) and the sets $\{i\kappa_j\}, \{\bar{\gamma}_j\}, \{\bar{b}_j\}$. Then, turning on the dynamics

$$\begin{aligned} \dot{\gamma}(t) &= -4ik^3\gamma(t) \\ \dot{\bar{b}}(t) &= -8ik^3\bar{b}(t) \\ \dot{\bar{a}}(t) &= 0 \end{aligned}$$

we can restore the potentials, using the equations (43),(44).

3 Appendix

3.1 Grassmann Algebra

Definition [13],[14] The finite dimensional Grassmann algebra $\Lambda(n)$ of order n is an algebra with n generators: $1, \Theta_1, \dots, \Theta_n$, satisfying the anticommutativity property $\Theta_i\Theta_j + \Theta_j\Theta_i = 0$. The element of this algebra is called even (odd), if in the decomposition of this element

$$\eta = \sum_{m \geq 0} \sum_{i_1 < \dots < i_m} \eta_{i_1 \dots i_m} \Theta^{i_1} \dots \Theta^{i_m}$$

each number m is even (odd). As vector space $\Lambda(n)$ splits into two subspaces $\Lambda(n) = \Lambda(n)_{\bar{0}} \oplus \Lambda(n)_{\bar{1}}$, where $\Lambda(n)_{\bar{0}}$ is a linear space of even elements and $\Lambda(n)_{\bar{1}}$ is a linear space of odd elements.

3.2 Lie superalgebra $\mathfrak{osp}(1|2)$

Definition [13], [14] Lie superalgebra (or Z_2 -graded Lie algebra) is a real or complex Z_2 -graded space with fixed parity $\alpha : \mathcal{G} \rightarrow Z_2$, such that $\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}$, where $\alpha(\mathcal{G}_{\bar{0}}) = 0, \alpha(\mathcal{G}_{\bar{1}}) = 1$, where the bilinear map $[\cdot, \cdot]$ is defined, such that for homogeneous elements (that is elements, lying in $\mathcal{G}_{\bar{0}}$ or in $\mathcal{G}_{\bar{1}}$) the following properties are valid:

$$\alpha([x, y]) = \alpha(x) + \alpha(y), \tag{46}$$

$$[x, y] = (-1)^{\alpha(x)\alpha(y)+1}[y, x], \quad (47)$$

$$\begin{aligned} &[x, [y, z]](-1)^{\alpha(x)\alpha(z)} + [z, [x, y]](-1)^{\alpha(z)\alpha(y)} \\ &+ [y, [z, x]](-1)^{\alpha(y)\alpha(x)} = 0. \end{aligned} \quad (48)$$

As an example we consider Lie superalgebra $\text{osp}(1|2)$:

$$\begin{aligned} [h, X^\pm] &= \pm 2X^\pm & [h, v^\pm] &= \pm v^\pm & [X^+, X^-] &= h \\ [v^+, v^-] &= -h & [X^\pm, v^\mp] &= v^\pm & [v^\pm, v^\pm] &= \pm 2X^\pm \end{aligned}$$

v^\pm are odd; h, X^\pm are even.

Matrix realization:

$$\begin{aligned} h &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & v_- &= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & v_+ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ X^+ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & X^- &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

3.3 Supergroups. Supergroup $\text{osp}(1|2)$

To construct the supergroup by means of superalgebra, we need to consider the exponential map, as in usual case. The difference is the following: you need to attach to each generator an element of Grassmann algebra of the same parity (the index of the exponent should be even).

Example: supergroup $\text{Osp}(1|2)$ [13], [14]

This group is generated by the matrices (in the defining representations) of the following form:

$$u = \begin{pmatrix} a & \alpha & b \\ \beta & f & \delta \\ c & \gamma & d \end{pmatrix},$$

with $uJu^{st} = J$, where

$$\begin{aligned} J &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ u^{st} &= \begin{pmatrix} a & -\beta & c \\ \alpha & f & \gamma \\ b & -\delta & d \end{pmatrix}, \end{aligned}$$

is a supertransposed matrix. Supertrace of the matrix u is defined by the expression $f - a - d = \text{str} u$.

The properties of the supertrace:

$$\begin{aligned}\text{str}(MN) &= (-1)^{\alpha(M)\alpha(N)} \text{str}(NM) \\ \text{str}(M + N) &= \text{str} M + \text{str} N \\ \text{str}(M^{st}) &= \text{str} M\end{aligned}$$

3.4 The notations, used in the text

We use the following notations: the Greek letters denote odd elements of Grassmann algebra, the Latin letters denote even elements.

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